Inventory Control
in a Decentralized Two-Stage
Make-to-Stock Queuing System

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Presentation outline

- Introduction
- Analysis of a two-stage supply-chain
- Two-stage make-to-stock queuing system
- Stackelberg game
  - Supplier’s problem
  - Manufacturer’s problem
- Conclusions
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Introduction

- **An enterprise network** consists of different companies that interact to produce families of goods.
  - Each member company seeks to optimize his production and inventory policy to maximize his own profit.
  - The objectives of the partners are generally antagonistic and can lead to contradictory choices and a global loss of economic efficiency.

- **Decisions in a supply chain: a game theory problem**
  - **Different enterprises (players)**
    - share common information and logistic networks
    - have full decisional autonomy
    - separate constraints
    - conflicting objectives
  - **Coordination contracts allow**
    - sharing risks, profits, or costs
    - obtaining globally optimal values as a Nash equilibrium
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Analysis of a two-stage supply-chain

Two-stage supply-chain facing demand and lead time uncertainties:

- Two-stage production/inventory system:
  - An end-product manufacturer
  - A component supplier

- Each stage of the supply chain:
  - carries a local finished-goods inventory to serve the demand
  - replenishes this inventory through a local production process

- Demand and processing times of the units are random variables.
Analysis of a two-stage supply-chain

- Because of random effects, all customer demands cannot be satisfied immediately.
- The customer service level is influenced by the inventory control decisions of both firms.
- Each firm is an individual profit maximizing entity:
  - The inventory control decisions may deviate from the system optimal solution.
- Determine a contracting scheme:
  - Setting the rules of trade and delivery between the two partners
  - Achieving system optimal performance
Analysis of a two-stage supply-chain

Hypothesis:

♦ The customers arrive at the manufacturer in accordance with a Poisson process having rate $\lambda$.

♦ The successive processing times of the units are independent exponential random variables with rate $\mu_i$ ($\mu_1 \neq \mu_2$).

♦ To control inventory, each firm is supposed to use a base-stock policy ($S_i-1 , S_i$).
Analysis of a two-stage supply-chain

Hypothesis:

- The customers arrive at the manufacturer in accordance with a Poisson process having rate $\lambda$.
- The successive processing times of the units are independent exponential random variables with rate $\mu_i$ ($\mu_1 \neq \mu_2$).
- To control inventory, each firm is supposed to use a base-stock policy $(S_i - 1, S_i)$.
  - $S_i$: base-stock level
  - The inventory initially contains $S_i \geq 0$ units.
  - A unitary replenishment order is placed whenever a demand occurs.

$\text{Inventory level} = S_i - 1$

$\text{Inventory Position} = \text{Inventory level} + \text{Outstanding orders} = S_i$
Literature Review

Game theoretic analyses:

- Caldentey and Wein (2003): The supplier operates in a make-to-order manner and controls the production rate. The retailer carries finished-goods inventory and determines the base-stock level.

- Jemai and Karaesmen (2004): The supplier operates in a make-to-stock manner with a fixed mean production rate. Each stage determines the local base-stock level.

- Gupta and Weerawat (2004): Each stage operates in a make-to-stock manner. The second stage base-stock level is given.
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Two-stage make-to-stock queuing system

Steady state random variables:

$P_i$: number of uncompleted orders waiting in the processing queue

$K_i$: number of uncompleted orders in the system ($P_i$ plus the order eventually in service)

$I_i$: number of units in the finished-goods inventory

$B_i$: number of outstanding backorders
Two-stage make-to-stock queuing system

Inventory Position = Inventory level + Outstanding orders

\[
\begin{align*}
S_i & : \text{Inventory position} \\
I_i - B_i & : \text{Inventory level} \\
K_i & : \text{Outstanding orders}
\end{align*}
\]

\[
\begin{align*}
S_i &= I_i - B_i + K_i \\
I_i &= (S_i - K_i)^+ \\
B_i &= (K_i - S_i)^+
\end{align*}
\]

\[
\begin{align*}
P_{k_i} &= \Pr\{K_i = k_i\} \\
E[I_i] &= \sum_{k_i=0}^{S_i} (S_i - k_i)P_{k_i} \\
E[B_i] &= \sum_{k_i=S_i}^{\infty} (k_i-S_i)P_{k_i}
\end{align*}
\]
Two-stage make-to-stock queuing system

\[ \Pr \left\{ K_1 = k_1 \right\} : \]

- The production facility of the supplier behaves as an M/M/1 queue with the traffic intensity \( \rho_1 = \frac{\lambda}{\mu_1} \).

- \( L_1 = W_1 \Rightarrow K_1 = N_1 \)

  \( L_1 \): lead time of the finished-goods inventory of stage 1

  \( W_1 \): sojourn time of an M/M/1 queue with traffic intensity \( \rho_1 \)

  \( K_1 \): number of uncompleted orders in the system

  \( N_1 \): number of units in an M/M/1 queue with traffic intensity \( \rho_1 \)

- \( P_{k_1} = \rho_1^{k_1}(1 - \rho_1) \Rightarrow E[B_1] = \frac{\rho_1^{S+1}}{1 - \rho_1} \)

  \( E[I_1] = S_1 - \frac{\rho_1(1 - \rho_1^{S_1})}{1 - \rho_1} \)
Pr \{ K_2 = k_2 \} - Approximation method of Lee and Zipkin (1992):

- The departure distribution of units from stage 1 is approximated as a Poisson process.

- \( L_2 \approx D_1 + W_2 \)
  
  \( L_2 \) : lead time of the finished-goods inventory of stage 2
  
  \( D_1 \) : delay for demands placed upon stage 1
  
  \( W_2 \) : sojourn time of an M/M/1 queue with traffic intensity \( \rho_2 \)

- \( L_2 \) has a continuous phase-type distribution:

\[
\text{Pr}\{ I_1 > 0 \} = 1 - \rho_1^{S_i} \\
\text{Pr}\{ I_1 = 0 \} = \rho_1^{S_i} \\
\mu_1\lambda \rightarrow \mu_2\lambda \\
\text{Phase 1} \rightarrow \text{Phase 2}
\]

- The probability distribution function of \( L_2 \):
  \[
f_{L_2}(t) = \rho_1^{S_i} f_{W_1 + W_2}(t) + (1 - \rho_1^{S_i}) f_{W_2}(t)
\]
Two-stage make-to-stock queuing system

Pr \{ K_2 = k_2 \} - Approximation method of Lee and Zipkin (1992):

\[
f_{L_2}(t) = \left( \frac{1 - \rho_2}{\rho_1 - \rho_2} \right)(\mu_1 - \lambda)e^{-(\mu_1 - \lambda)t} + \left( 1 - \frac{1}{\rho_1} \right) \left( 1 - \frac{1}{\rho_2} \right) (\mu_2 - \lambda)e^{-(\mu_2 - \lambda)t}
\]

- The number of outstanding orders in the system has the same distribution as the number of demands in a random time \( L_2 \):

\[
P_{k_2} = \sum_{t=0}^{\infty} f_{L_2}(t) e^{-\lambda t} \frac{(\lambda t)^{k_2}}{k_2!} dt = \left( \frac{1}{\rho_1} \right) \left( 1 - \frac{1}{\rho_1} \right) \left( 1 - \frac{1}{\rho_2} \right) (1 - \rho_2)
\]

\[
\Rightarrow E[B_2] = \left( \frac{1 - \rho_2}{\rho_1 - \rho_2} \right) \left( \frac{1}{\rho_1} \right) + \left( 1 - \frac{1}{\rho_1} \right) \left( \frac{1 - \rho_2}{\rho_1 - \rho_2} \right) \left( 1 - \frac{1}{\rho_2} \right)
\]

\[
E[I_2] = S_2 - \frac{1}{\rho_1} - \frac{1}{\rho_2} + \left( \frac{1}{\rho_1} \right) \left( \frac{1 - \rho_2}{\rho_1 - \rho_2} \right) \left( 1 - \frac{1}{\rho_1} \right) + \left( 1 - \frac{1}{\rho_1} \right) \left( \frac{1 - \rho_2}{\rho_1 - \rho_2} \right) \left( 1 - \frac{1}{\rho_2} \right)
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Stackelberg game

Theoretical case of a centralized system:

\[
\max_{S_1, S_2 \geq 0} \pi_0(S_1, S_2) = \pi_1(S_1) + \pi_2(S_1, S_2) = \lambda(p_2 - c_1 - c_2) - h_1 E[I_1] - h_2 E[I_2] - b_2 E[B_2]
\]

- \( p_2 \) : purchasing price per unit of end-product
- \( b_2 \) : backorder cost per customer backorder per time unit
- \( c_i \) : production cost per unit produced
- \( h_i \) : holding cost per unit of inventory per time unit

Decentralized system:

- Each firm attempts to maximize his own steady state expected profit per time unit.
- The main operational decision at each stage is the base-stock level \( S_i \).
- The supplier lacks the incentive to implement \( S_1 > 0 \).
- The decentralized system should be equipped with a coordination mechanism.
**Stackelberg game**

*Stackelberg game:*

- The manufacturer is the leader and the supplier is the follower.
- The manufacturer offers to the supplier a two-part linear contract \((p_1, b_1)\):
  - \(p_1\): purchasing price per unit of component
  - \(b_1\): backorder penalty per component backorder per time unit

**Expected rate of the transfer payment**

\[
T(S_1, p_1, b_1) = \lambda p_1 - b_1 E[B_1]
\]

\[
\pi_1(S_1, p_1, b_1) = T(S_1, p_1, b_1) - \lambda c_1 - h_1 E[I_1]
\]

**Steady state expected profit rate of the manufacturer:**

\[
\pi_2(S_1, S_2, p_1, b_1) = \lambda (p_2 - c_2) - h_2 E[I_2] - b_2 E[B_2] + T(S_1, p_1, b_1)
\]
Supplier’s problem

Steady state expected profit rate of the supplier:

\[ \pi_1(S_1, p_1, b_1) = \lambda (p_1 - c_1) - h_i E[I_1] - b_i E[B_1] = \lambda (p_1 - c_1) - h_i \left( S_1 - \frac{\rho_1}{1 - \rho_1} \right) - (h_i + b_1) \frac{\rho_1^{S_1+1}}{1 - \rho_1} \]

Optimization problem:

\[
\max_{S_1 \geq 0} \pi_1(S_1, p_1, b_1)
\]

- \( S_1 \) is a continuous variable.
- The profit function of the supplier
  - is concave in \( S_1 \)
  - is decreasing in \( S_1 \) if \( b_1 < b_1^{\text{min}} \) where \( b_1^{\text{min}} = -h_i \left( 1 + \frac{1 - \rho_1}{\rho_1 \ln \rho_1} \right) > 0 \)

Optimal value of the base-stock level \( S_1 \):

\[ S_1^*(b_1) = \begin{cases} 
\ln \left( \frac{-h_i (1 - \rho_1)}{(h_i + b_1) \rho_1 \ln \rho_1} \right) / \ln \rho_1 & \text{if } b_1 > b_1^{\text{min}} \\
0 & \text{if } b_1 \leq b_1^{\text{min}} 
\end{cases} \]

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Manufacturer’s problem

Steady state expected profit rate of the manufacturer:

\[ \pi_2(S_1, S_2, p_1, b_1) = \lambda(p_2 - p_1 - c_2) - h_2 \left( S_2 - \rho_2 \frac{\rho_1^{S_1+1}}{1 - \rho_1} \right) + b_1 \frac{\rho_1^{S_1+1}}{1 - \rho_1} - (h_2 + b_2) \frac{\rho_2^{S_2+1}}{1 - \rho_2} \]

\[ - (h_2 + b_2) \left( \frac{\rho_1^{S_1+1}}{\rho_1 - \rho_2} \right) \left( \frac{\rho_1^{S_2+1}}{1 - \rho_1} - \frac{\rho_2^{S_2+1}}{1 - \rho_2} \right) \]

Optimization problem:

\[ \max_{p_1, b_1, S_1 \geq 0} \pi_2(S_1^*, S_2, p_1, b_1) \]

s.t. \[ \pi_1(S_1^*(b_1), p_1, b_1) \geq 0 \quad \text{(individual rationality constraint)} \]
Manufacturer’s problem

Optimal value of the purchasing price $p_1$:

- $L = \pi_2(S^*_1(b_1), S_2, p_1, b_1) + u\pi_1(S^*_1(b_1), p_1, b_1) \Rightarrow u^* = 1$

- The optimal value of the purchasing price satisfies $\pi_1(S^*_1(b_1), p_1^*(b_1), b_1) = 0$:

$$p^*_1(b_1) = c_1 + \frac{h_1}{\lambda} \left( S^*_1(b_1) - \frac{\rho_1}{1 - \rho_1} \right) + \frac{h_1 + b_1}{\lambda} \left( \frac{\rho_1^{S^*_1(b_1)+1}}{1 - \rho_1} \right)$$

- The contract $(p_1^*(b_1), b_1)$ transfers all the operating costs and benefits of the supplier to the manufacturer letting the supplier with zero profit.
Manufacturer’s problem

Optimal value of the backorder penalty $b_1$:

- Let $\pi_2(S_2, b_1)$ denote $\pi_2(S_1^*(b_1), S_2, p_1^*(b_1), b_1)$.

  - For $b_1 \leq b_1^{\text{min}}$, profit function $\pi_2(S_2, b_1)$ is constant.
  
  - For $b_1 > b_1^{\text{min}}$, profit function $\pi_2(S_2, b_1)$:
    
    - is strictly quasi-concave if $S_2 < S_2^{\text{max}}$.
    
    - is strictly decreasing if $S_2 \geq S_2^{\text{max}}$.

- Given $S_2$, the optimal value of the backorder penalty is

  $$
  b_1^*(S_2) = \begin{cases} 
  (h_2 + b_2)\tau_0(S_2) - h_2 & \text{if } S_2 < S_2^{\text{max}} \\
  b_1^{\text{min}} & \text{if } S_2 \geq S_2^{\text{max}} 
  \end{cases}
  $$

where

$$
\tau_0(S_2) = \frac{(1 - \rho_1)(1 - \rho_2)}{\rho_1 - \rho_2} \left( \frac{\rho_1^{S_2} + 1}{1 - \rho_1} - \frac{\rho_2^{S_2} + 1}{1 - \rho_2} \right)
$$

and

$$
\tau_0(S_2^{\text{max}}) = \frac{(h_2 + b_1^{\text{min}})}{(h_2 + b_2)}
$$
**Manufacturer’s problem**

*Optimal contract parameters:*

- **If** $S_2 < S_2^{\text{max}}$ : 
  - **The manufacturer** offers the contract: $(p_1^*(b_1^*), b_1^* > b_1^{\text{min}})$
  - **The supplier** installs the base-stock level: $S_1^*(b_1^*) > 0$

- **If** $S_2 \geq S_2^{\text{max}}$ : 
  - **The manufacturer** offers the contract: $(p_1^*(b_1^*), \ b_1^* = b_1^{\text{min}})$
  - **The supplier** installs the base-stock level: $S_1^*(b_1^{\text{min}}) = 0$

- **The manufacturer** obtains the **total expected profit of the decentralized system**.
- **The supplier** obtains **zero profit**.
Manufacturers's problem

Optimal value of the base-stock level $S_2$:

- Let $\pi_2(S_2)$ denote $\pi_2(S_2, b_1^*(S_2))$.
- Profit function $\pi_2(S_2)$ is strictly quasi-concave for $S_2 \geq 0$ under certain conditions that depend on the problem parameters.
- The optimal value of the base-stock level is

$$S_2^* = \begin{cases} S_{2}^{\text{opt}_1} & \text{if } \tau_2(S_2^\text{max}) > -h_2/(h_2 + b_2) \\ S_{2}^{\text{opt}_2} & \text{if } \tau_2(S_2^\text{max}) \leq -h_2/(h_2 + b_2) \end{cases}$$

where

$$\tau_1(S_2^{\text{opt}_1}) = \frac{-h_2}{h_2 + b_2}, \quad \tau_2(S_2^{\text{opt}_2}) = \frac{-h_2}{h_2 + b_2}$$

$$\tau_2(S_2) = \left( \frac{\rho_1 (1 - \rho_2)}{\rho_1 - \rho_2} \right) \rho_1^{S_2^2 + 1} \ln \rho_1 + \left( 1 - \frac{\rho_1 (1 - \rho_2)}{\rho_1 - \rho_2} \right) \rho_2^{S_2^2 + 1} \ln \rho_2$$

$$\tau_1(S_2) = \left( \frac{-h_1 (1 - \rho_1)(1 - \rho_2)}{(h_1 + b_1) \ln \rho_1 (\rho_1 - \rho_2)} \right) \rho_1^{S_2^2 + 1} \ln \rho_1 + \left( 1 - \frac{-h_1 (1 - \rho_1)(1 - \rho_2)}{(h_1 + b_1) \ln \rho_1 (\rho_1 - \rho_2)} \right) \rho_2^{S_2^2 + 1} \ln \rho_2$$
**Manufacturer’s problem**

*Evolution of \( S_1^* (b_1^* (S_2^*)) \) and \( S_2^* \): \( h_2 = 0.8 \), \( b_2 = 10 \), \( \rho_2 = 0.5 \)*

- \( h_1 = 0.6 \)
- \( h_1 = 0.2 \)

The contract \((p_1^*(b_1^*),b_1^*)\) transfers all the operating costs and benefits of the supplier to the manufacturer.
### System optimal solution

- **Optimization problem of the centralized system:**

\[
\max_{s_1, s_2 \geq 0} \pi_0(s_1, s_2) = \pi_1(s_1, p_1, b_1) + \pi_2(s_1, s_2, p_1, b_1) = \lambda(p_1 - c_1 - c_2) - h_1E[I_1] - h_2E[I_2] - b_2E[B_2]
\]

\[
S_1^0(s_2) = \arg\max_{s_1 \geq 0} \pi_0(s_1, s_2)
\]

\[
S_2^0 = \arg\max_{s_2 \geq 0} \pi_0(S_1^0(s_2), s_2)
\]

- The system optimal solution is the same as the Stackelberg equilibrium:

\[
S_1^*(b_1^*(s_2)) = S_1^0(s_2) \quad \pi_0^* = \pi_1^* + \pi_2^*
\]

\[
S_2^* = S_2^0
\]

- Offering \((p_1^*(b_1^*), b_1^*)\) the *manufacturer obtains the integrated system profit* and lets the supplier with his reservation profit.
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Conclusions

- In enterprise networks, decentralized decisions are generally less efficient than a centralized mechanism maximizing a global profit function.

- Contracts between partners
  - strengthen commitments of the partners through risk, profit, or cost sharing using a transfer payment scheme,
  - create incentives to implement globally optimal actions.

- **Coordination contract**: a performance-based pricing scheme
  - Component purchasing price
  - Component late-delivery penalty

- The whole system is globally optimized with respect to the inventory capacities.

- In a Stackelberg game, the dominant player leads the game:
  - The manufacturer captures the maximal global profit of the supply chain.
  - The supplier is maintained at his minimal acceptable satisfaction level.
  - The manufacturer may jointly optimize his base-stock level and the contract parameters.